

# QUASI-STATIONARY DISTRIBUTIONS ASSOCIATED WITH EXPLOSIVE CSBPs

Cyril Labbé

*Laboratoire de Probabilités et Modèles Aléatoires*

*Université Pierre et Marie Curie*

cyril.labbe@upmc.fr

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## Abstract

We characterize all the quasi-stationary distributions and the  $Q$ -process associated with a continuous state branching process that explodes in finite time. We also provide a rescaling for the continuous state branching process conditioned on non-explosion when the branching mechanism is regularly varying at 0.

## 1 Introduction

Consider a continuous-state branching process (CSBP for short)  $t \mapsto Z_t(x)$  starting from the value  $x \geq 0$  and whose branching mechanism is given by the convex function

$$\Psi(u) = \gamma u + \frac{\sigma^2}{2} u^2 + \int_{(0,\infty)} (e^{-uh} - 1 + uh \mathbf{1}_{\{h < 1\}}) \nu(dh), \quad \forall u \geq 0 \quad (1)$$

where  $\gamma \in \mathbb{R}$ ,  $\sigma \geq 0$  and  $\nu$  is a Borel measure on  $(0, \infty)$  such that  $\int_{(0,\infty)} (1 \wedge h^2) \nu(dh) < \infty$ . It is well known that, at any given time  $t \geq 0$ ,  $x \mapsto Z_t(x)$  is a subordinator: we denote by  $u(t, \cdot)$  its Laplace exponent. In [8], Silverstein showed that the drift  $d_t$  of this subordinator equals 0 when the CSBP has infinite variation paths. The following result gives the expression of this drift when it is not 0.

**Proposition 1.1** *When  $\Psi$  is the Laplace exponent of a Lévy process with infinite variation paths,  $d_t = 0$ . Otherwise, for all  $t \geq 0$  we have*

$$d_t = e^{-Dt}, \quad \text{where } D := \gamma + \int_{(0,1)} h \nu(dh) \quad (2)$$

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This entails that  $d_t$  only depends on the drift of the Lévy process whose Laplace exponent is given by  $\Psi$ .

The goal of this paper is to investigate the quasi-stationary distributions associated with a CSBP that explodes in finite time almost surely. From now on,  $\mathbb{P}_x$  is the distribution on  $\mathbb{D}([0, +\infty), [0, +\infty])$  of a CSBP starting from  $x$ , and  $Z$  is the canonical process. More generally, given a probability measure  $\mu$  on  $[0, \infty)$  we denote by  $\mathbb{P}_\mu$  the law of the CSBP initially distributed according to  $\mu$ . Let  $T$  be the lifetime of  $Z$ , that is, the random time defined by  $T := T_0 \wedge T_\infty$  where

$$\begin{aligned} (\text{Extinction}) \quad T_0 &:= \inf\{t \geq 0 : Z_t = 0\} \\ (\text{Explosion}) \quad T_\infty &:= \inf\{t \geq 0 : Z_t = \infty\} \end{aligned}$$

By quasi-stationary distribution (QSD for short), we mean a probability measure  $\mu$  on  $(0, \infty)$  such that

$$\mathbb{P}_\mu(Z_t \in \cdot \mid T > t) = \mu(\cdot)$$

In that case, it is a simple matter to check that under  $\mathbb{P}_\mu$  the random variable  $T$  has an exponential distribution, the parameter of which is called the *rate of decay* of  $\mu$ .

*A brief review of the litterature: the extinction case.*

Li [7] and Lambert [5] considered the extinction case  $T = T_0 < \infty$  almost surely and studied the CSBP conditioned on non-extinction. Let us recall some of their results. When  $\Psi$  is subcritical, that is  $\Psi'(0) > 0$ , there exists a family  $(\mu_\beta; 0 < \beta \leq \Psi'(0))$  of quasi-stationary distributions on  $(0, \infty)$  where  $\beta$  is the rate of decay of  $\mu_\beta$ . These distributions are characterized by their Laplace transforms as follows

$$\int_{(0, \infty)} \mu_\beta(dr) e^{-r\lambda} = 1 - e^{-\beta\Phi(\lambda)}, \quad \forall \lambda \geq 0 \quad (3)$$

where  $\Phi(\lambda) := \int_\lambda^{+\infty} du/\Psi(u)$ . For any  $\beta > \Psi'(0)$  they proved that there is no QSD with rate of decay  $\beta$ , and that Equation (3) does not define the Laplace transform of a probability measure on  $(0, \infty)$ . Additionally, the value  $\beta = \Psi'(0)$  yields the so-called Yaglom limit:

$$\mathbb{P}_x(Z_t \in \cdot \mid T > t) \xrightarrow[t \rightarrow \infty]{} \mu_{\Psi'(0)}(\cdot), \quad \forall x > 0$$

When  $\Psi$  is critical, that is  $\Psi'(0) = 0$ , the preceding quantity converges to a trivial limit for all  $x > 0$  and Equation (3) does not define the Laplace transform of a probability measure on  $(0, \infty)$ . However, under the condition  $\Psi''(0) < \infty$ , they proved the following convergence

$$\mathbb{P}_x\left(\frac{Z_t}{t} \geq z \mid T > t\right) \xrightarrow[t \rightarrow \infty]{} \exp\left(-\frac{2z}{\Psi''(0)}\right), \quad \forall x > 0, z \geq 0 \quad (4)$$

In Section 3, we will provide a generalisation of this result only assuming that  $\Psi$  is regularly varying at 0, see the forthcoming Proposition 3.1.

Finally in both critical and subcritical cases, for any given value  $t > 0$  the process  $(Z_r, r \in [0, t])$  conditioned on  $s < T$  admits a limiting distribution as  $s \rightarrow \infty$ , called  $Q$ -process. The law of the  $Q$ -process is obtained as a  $h$ -transform of  $\mathbb{P}$  as follows

$$d\mathbb{Q}_{x|\mathcal{F}_t} := \frac{Z_t e^{\Psi'(0)t}}{x} d\mathbb{P}_{x|\mathcal{F}_t}, \quad \forall x > 0$$

*Main results: the explosive case.*

Suppose  $T = T_\infty < \infty$  almost surely and remark that conditioning a CSBP on non-explosion does not affect the branching property. Subsequently the law of  $Z_t$  conditioned on  $T > t$  is infinitely divisible: if it admits a limit as  $t$  goes to  $\infty$ , the limit has to be infinitely divisible as well.

In [4], Grey proved that the almost sure explosion is equivalent with

$$\Psi(u) \leq 0, \forall u \geq 0 \text{ and } \int_{0+} \frac{du}{\Psi(u)} < \infty \quad (5)$$

In particular, the first condition ensures that  $\Psi$  is the Laplace exponent of a subordinator. We introduce the notation:

$$\Psi(+\infty) := \lim_{u \rightarrow \infty} \Psi(u) \in [-\infty, 0)$$

It is a simple matter to verify that

$$\Psi(+\infty) = -\infty \iff \nu(0, \infty) = \infty \text{ or } D < 0$$

Conversely,  $\Psi(+\infty) \in (-\infty, 0)$  implies  $\Psi(+\infty) = -\nu(0, \infty)$ . Our result below shows that this quantity plays a rôle analogue to  $\Psi'(0+)$  in the extinction case.

**Theorem 1** *Suppose  $T = T_\infty < \infty$  almost surely and set*

$$\Phi(\lambda) := \int_{\lambda}^0 \frac{du}{\Psi(u)}, \forall \lambda \geq 0$$

*For any  $\beta > 0$  there exists a unique quasi-stationary distribution  $\mu_\beta$  associated to the rate of decay  $\beta$  which is infinitely divisible and is characterized by*

$$\int_{(0, \infty)} \mu_\beta(dr) e^{-r\lambda} = e^{-\beta\Phi(\lambda)}, \forall \lambda \geq 0 \quad (6)$$

*Additionally, the following dichotomy holds true:*

(i)  $\Psi(+\infty) \in (-\infty, 0)$ . *The limiting conditional distribution is given by*

$$\lim_{t \rightarrow \infty} \mathbb{P}_x(Z_t \in \cdot \mid T > t) = \mu_{x\nu(0, \infty)}(\cdot), \forall x \in (0, \infty)$$

(ii)  $\Psi(+\infty) = -\infty$ . *The limiting conditional distribution is trivial:*

$$\lim_{t \rightarrow \infty} \mathbb{P}_x(Z_t \leq a \mid T > t) = 0, \forall a, x \in (0, \infty)$$

Let us make some comments. Firstly this theorem implies that  $\lambda \mapsto \Phi(\lambda)$  is the Laplace exponent of a subordinator, and so,  $\mu_\beta$  is the distribution of a  $\Phi$ -Lévy process taken at time  $\beta$ . Secondly there is a similarity with the extinction case: the limiting conditional distribution is trivial i.f.f.  $\Psi(+\infty) = -\infty$  so that the dichotomy on the value  $\Psi(+\infty)$  is the explosive counterpart of the dichotomy on the value  $\Psi'(0)$  in the extinction case. Also, note the similarity in the definition of the Laplace transforms (3) and (6). However, there are two major differences with the extinction case: firstly there is no restriction on

the rates of decay. Secondly, even if the limiting conditional distribution is trivial when  $\Psi(+\infty) = -\infty$ , there exists a family of QSDs. We now propose, when  $\Psi$  is regularly varying at 0, a rescaling of the CSBP conditioned on non-explosion that ensures a non-trivial limit. Recall that we call slowly varying function at 0 any continuous map  $L : (0, \infty) \rightarrow (0, \infty)$  such that for any  $a \in (0, \infty)$ ,  $L(au)/L(u) \rightarrow 1$  as  $u \downarrow 0$ .

**Theorem 2** *Suppose that  $\Psi(u) = -u^{1-\alpha}L(u)$  with  $L$  a slowly varying function at 0 and  $\alpha \in (0, 1)$ , and assume that  $\Psi(+\infty) = -\infty$ . Consider any function  $f : [0, \infty) \rightarrow (0, \infty)$  satisfying  $\Psi(f(t)^{-1})f(t) \sim \Psi(u(t, 0+))$  as  $t \rightarrow \infty$ . Then the following convergence holds true:*

$$\mathbb{E}_x \left[ e^{-\lambda Z_t / f(t)} \mid t < T \right] \xrightarrow[t \rightarrow \infty]{} e^{-x \lambda^\alpha / \alpha}, \quad \forall x, \lambda \in (0, \infty)$$

Notice that such a function  $f$  exists. Indeed, the map  $u \mapsto \Psi(u)/u$  is strictly increasing and takes all values in  $(-\infty, D)$ , the existence then follows from simple arguments.

**Example 1.2** *When  $\Psi(u) = -k u^{1-\alpha}$  with  $k > 0$  and  $\alpha \in (0, 1)$ , we have  $f(t) \sim (\alpha k t)^{(1-\alpha)/\alpha^2}$  as  $t \rightarrow \infty$ . When  $\Psi(u) = -c u - k u^{1-\alpha}$  with  $k, c > 0$  and  $\alpha \in (0, 1)$ , we have  $f(t) \sim (k/c)^{(1-\alpha)/\alpha^2} e^{ct/\alpha}$  as  $t \rightarrow \infty$ .*

Finally, we compute the  $Q$ -process associated with an explosive CSBP. Let  $\mathcal{F}_t$  be the sigma-field generated by  $(Z_s, s \in [0, t])$ , for any  $t \in [0, \infty)$ .

**Theorem 3** *We only assume that  $T = T_\infty < \infty$  almost surely. For each  $x > 0$ , there exists a distribution  $\mathbb{Q}_x$  on  $\mathbb{D}([0, \infty), [0, \infty))$  such that for any  $t \geq 0$  and any event  $\Theta \in \mathcal{F}_t$*

$$\lim_{s \rightarrow \infty} \mathbb{P}_x(\Theta \mid T > s) = \mathbb{Q}_x(\Theta)$$

*Furthermore,  $\mathbb{Q}_x$  is the law of the  $\Psi^Q$ -CSBP where*

$$\Psi^Q(u) = Du$$

The  $Q$ -process appears as the  $\Psi$ -CSBP from which one has removed all the jumps: only the deterministic part remains (recall the expression of the drift from Proposition 1.1). Notice that the  $Q$ -process cannot be defined through a  $h$ -transform of the CSBP: actually the distribution of the  $Q$ -process on  $\mathbb{D}([0, t], [0, \infty))$  is not even absolutely continuous with respect to that of the  $\Psi$ -CSBP, except when the Lévy measure  $\nu$  is finite.

The paper is organized as follows. In the second section, we examine the discrete case: as no explosion occurs in discrete time, we need to consider continuous-time Galton-Watson processes. We provide a complete description of the QSDs when this process explodes in finite time almost surely. In the third section, we recall basic facts about CSBPs then we prove Propositions 1.1 and 3.1. In the fourth section, we prove Theorems 1, 2 and 3.

## 2 The discrete case

A discrete-state branching process  $(Z_t, t \geq 0)$  is a continuous-time Markov process taking values in  $\mathbb{Z}_+$  that verifies the branching property (see for instance [1], Chapter III). It can be seen as a Galton-Watson process with offspring distribution  $\xi$  where each individual has an independent exponential lifetime with parameter  $c > 0$ . Let us denote by  $\phi(\lambda) = \sum_{k=0}^{\infty} \lambda^k \xi(k)$ ,  $\forall \lambda \in [0, 1]$  the generating function of the Galton-Watson process. We denote by  $\mathbf{P}_n$  the law on the space  $\mathbb{D}([0, \infty), \mathbb{Z}_+)$  of  $Z$  starting from  $n \in \mathbb{Z}_+$ , and  $\mathbf{E}_n$  the related expectation operator. More generally, we denote by  $\mathbf{P}_\mu$  the law of  $Z$  with initial distribution  $\mu$ , where  $\mu$  is a probability distribution on  $\mathbb{Z}_+$ . The semigroup of the DSBP is characterized via the Laplace transform

$$\forall r \in (0, 1), \forall t \in [0, \infty), \quad \mathbf{E}_n[r^{Z_t}] = F(t, r)^n \text{ where } \int_r^{F(t, r)} \frac{dx}{c(\phi(x) - x)} = t \quad (7)$$

We denote by  $\tau$  the lifetime of  $Z$ , that is, the infimum of the extinction time  $\tau_0$  and the explosion time  $\tau_\infty$ : it is well-known that  $\mathbf{P}_n(\tau_0 < t) = F(t, 0+)^n$  and  $\mathbf{P}_n(\tau_\infty < t) = 1 - F(t, 1-)^n$ . In this section, we assume that there is explosion in finite time almost surely: this implies that the smallest solution to the equation  $\phi(x) = x$  equals 0 and that  $\int_{1-} \frac{dx}{c(\phi(x) - x)} < \infty$ . We define

$$\Phi(r) := \int_1^r \frac{dx}{c(\phi(x) - x)}, \quad r \in (0, 1]$$

Clearly  $r \mapsto \Phi(r)$  is the inverse function of  $t \mapsto F(t, 1-)$ , that is for all  $t \geq 0$ ,  $\Phi(F(t, 1-)) = t$ . We say that a measure  $\mu$  on  $\mathbb{N} = \{1, 2, \dots\}$  is a quasi-stationary distribution (QSD) for  $Z$  if

$$\mathbf{P}_\mu(Z_t \in \cdot \mid \tau > t) = \mu(\cdot)$$

From the Markov property, we deduce that  $\tau$  has an exponential distribution under  $\mathbf{P}_\mu$ , the parameter of which is called the rate of decay of  $\mu$ .

**Theorem 4** *Suppose there is explosion in finite time almost surely. Let  $\beta_0 := c(1 - \xi(1))$ . There is a unique quasi-stationary distribution  $\mu_\beta$  associated with the rate of decay  $\beta$  if and only if  $\beta$  is of the form  $n\beta_0$ , with  $n \in \mathbb{N}$ . It is characterized by its Laplace transform*

$$\sum_k \mu_\beta(k) r^k = e^{-\beta \Phi(r)}, \quad \forall r \in (0, 1]$$

For any initial condition  $n \in \mathbb{N}$  we have

$$\lim_{t \rightarrow \infty} \mathbf{P}_n(Z_t \in \cdot \mid \tau > t) = \mu_{n\beta_0}(\cdot)$$

Finally the  $Q$ -process is the constant process. More precisely denote by  $\mathcal{F}_t$  the sigma-field generated by  $(Z_s, s \in [0, t])$  for any  $t \geq 0$ . For any event  $\Theta \in \mathcal{F}_t$  and any initial condition  $n \in \mathbb{N}$

$$\lim_{s \rightarrow \infty} \mathbf{P}_n(\Theta \mid \tau > s) = \mathbf{Q}_n(\Theta)$$

where  $\mathbf{Q}$  is the distribution of the DSBP with the trivial generating function  $F(t, r) = r$ .

Let us make some comments. First there exists only a countable family of QSDs. This is due to the restrictive condition that our process takes values in  $\mathbb{N}$ . Also, observe the similarity with Theorems 1 and 3: indeed a DSBP can be seen as a particular CSBP starting from an integer and whose branching mechanism is the Laplace transform of a compound Poisson process with integer-valued jumps. In particular  $\nu(\{k\}) = c\xi(k+1)$  for every integer  $k \geq 1$ . Hence the quantity  $c(1 - \xi(1))$  in the DSBP case corresponds to  $\nu(0, \infty)$  in the CSBP case.

*Sketch of the proof.* We only prove that  $\beta$  has to be of the form  $\beta_0 n$ , with  $n \in \mathbb{N}$ . The other properties derive from simple calculations. Fix  $n \in \mathbb{N}$  and observe that

$$\mathbb{E}_n[r^{Z_t} \mid \tau > t] \xrightarrow[t \rightarrow \infty]{} \exp(-\Phi(r)n\beta_0)$$

From this last convergence and the fact that  $\Phi(r) \xrightarrow[r \uparrow 1]{} 0$ , we deduce that  $r \mapsto \exp(-\Phi(r)n\beta_0)$  is the Laplace transform of a probability measure  $\mu_{\beta_0}$  on  $\mathbb{Z}_+$ . As  $\Phi(r) \xrightarrow[r \downarrow 0]{} +\infty$ , we deduce that this probability measure does not charge 0. Also, a simple calculation ensures that  $\mu_{\beta_0}(\{1\}) > 0$ .

Conversely, suppose that  $r \mapsto e^{-\beta\Phi(r)}$  is the Laplace transform of a probability measure on  $\mathbb{N}$ . Denote by  $m \in \mathbb{N}$  the smallest integer such that  $\mu_\beta(\{m\}) > 0$ . Then we have for all  $r \in (0, 1)$

$$\begin{aligned} e^{-\beta\Phi(r)} &= \mu_\beta(\{m\})r^m + \sum_{k>m} \mu_\beta(\{k\})r^k = (e^{-\beta_0\Phi(r)})^{\frac{\beta}{\beta_0}} \\ &= \left( \mu_{\beta_0}(\{1\})r + \sum_{k>1} \mu_{\beta_0}(\{k\})r^k \right)^{\frac{\beta}{\beta_0}} \end{aligned}$$

This implies that

$$\mu_\beta(\{m\})r^m \underset{r \downarrow 0}{\sim} (\mu_{\beta_0}(\{1\})r)^{\frac{\beta}{\beta_0}}$$

and so,  $m = \frac{\beta}{\beta_0} \in \mathbb{N}$ . ■

### 3 Continuous-state branching processes, drift and critical case

Recall the form taken by a branching mechanism  $\Psi$  from (1). A continuous-state branching process associated with  $\Psi$  is a Markov process whose semigroup is characterized by the following identity

$$\mathbb{E}_x[e^{-\lambda Z_t}] = e^{-x u(t, \lambda)}, \quad \forall \lambda > 0 \quad (8)$$

where

$$\partial_t u(t, \lambda) = -\Psi(u(t, \lambda)), \quad u(0, \lambda) = \lambda \quad (9)$$

We will denote by  $q$  the second root of  $\Psi$  that is  $q := \sup\{u \geq 0 : \Psi(u) \leq 0\}$ . Note that  $q$  is 0 in the subcritical and critical cases (*i.e.* when  $\Psi'(0) \geq 0$ ) while it equals  $\infty$  when  $\Psi$  is the Laplace exponent of a subordinator.

Classical results [3] ensure that  $\lambda \mapsto u(t, \lambda)$  is the Laplace exponent of a subordinator. Thanks to the Lévy-Khintchine formula, there exist  $a_t, d_t \geq 0$  and a Borel measure  $w_t$  on  $(0, \infty)$  with  $\int_{(0, \infty)} (1 \wedge h) w_t(dh) < \infty$  such that for all  $\lambda \in (0, \infty)$

$$u(t, \lambda) = a_t + d_t \lambda + \int_{(0, \infty)} (1 - e^{-\lambda h}) w_t(dh) \quad (10)$$

### 3.1 Proof of Proposition 1.1

For  $d_t$  to be positive it is necessary and sufficient that

$$\sigma = 0 \text{ and } \int_{(0,\infty)} (1 \wedge h) \nu(dh) < \infty$$

see Corollary in [8]. This is equivalent to saying that  $\Psi$  is the Laplace exponent of a Lévy process with finite variation paths. We suppose this condition fulfilled until the end of the proof. In this case, we can write

$$\Psi(u) = Du + \int_{(0,\infty)} (e^{-uh} - 1) \nu(dh)$$

Results [2] on the Laplace exponent of a Lévy process with finite variation paths ensure that

$$\frac{\Psi(u)}{u} \xrightarrow{u \rightarrow \infty} D, \quad \frac{u(t, \lambda)}{\lambda} \xrightarrow{\lambda \rightarrow \infty} d_t$$

Also, from the very definition of  $u(t, \lambda)$  one immediately gets for all  $t \geq 0, \lambda > 0$

$$\int_{u(t, \lambda)}^{\lambda} \frac{du}{\Psi(u)} = t \tag{11}$$

Now remark that  $u(t, \lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$  and write for any  $t \geq 0, \lambda > 0$

$$\log \left( \frac{u(t, \lambda)}{\lambda} \right) = \int_0^t \frac{\partial_s u(s, \lambda)}{u(s, \lambda)} ds = - \int_0^t \frac{\Psi(u(s, \lambda))}{u(s, \lambda)} ds$$

We have  $\Psi(u)/u \rightarrow D$  as  $u \rightarrow \infty$  and  $s \mapsto u(s, \lambda)$  is monotone: the dominated convergence theorem implies that  $\log(u(t, \lambda)/\lambda) \rightarrow -Dt$  as  $\lambda \rightarrow \infty$ . This ends the proof. ■

### 3.2 A more general result in the critical case with extinction

In [9], Slack showed that any critical Galton-Watson process with a regularly varying generating function can be properly rescaled so that, conditioned on non-extinction, it converges towards a non-trivial limit. In the present paper, we prove a similar result for critical CSBPs that become extinct almost surely.

**Proposition 3.1** *Suppose that  $\Psi(u) = u^{1+\alpha}L(u)$  with  $L$  a slowly varying function at 0 and  $\alpha \in (0, 1]$ . Assume that  $T = T_0 < \infty$  almost surely. Fix any function  $f : [0, \infty) \rightarrow (0, \infty)$  verifying  $f(t) \sim u(t, \infty)$  as  $t \rightarrow \infty$ . Then we have the following convergence*

$$\mathbb{E}_x \left[ e^{-\lambda Z_t f(t)} \mid t < T \right] \xrightarrow{t \rightarrow \infty} 1 - (1 + \lambda^{-\alpha})^{-1/\alpha}, \quad \forall x, \lambda \in (0, \infty)$$

We recover in particular the finite variance case (4) of Lambert and Li. Our result also covers the so-called stable branching mechanisms  $\Psi(u) = u^{1+\alpha}$  with  $\alpha \in (0, 1]$ . Once again one should notice the similarity between the explosion and the extinction cases: the assumption made in this proposition on the branching mechanism is quite similar to that of Theorem 2.

**Proof** The proof is similar to that of Theorem 1 in [9], so that some calculations are left to the reader. Recall that the extinction in finite time implies that for any  $t > 0$ ,  $u(t, \infty) < \infty$ . Fix  $x, \lambda \in (0, \infty)$ . First, remark that for any  $r \in (0, \infty)$  we have

$$\frac{1}{r u(r, \infty)^\alpha L(u(r, \infty))} = \frac{1}{r} \int_0^r \partial_s u(s, \infty) \frac{\Psi(u(s, \infty)) - u(s, \infty) \Psi'(u(s, \infty))}{\Psi(u(s, \infty))^2} ds$$

Since  $\Psi$  is regularly varying at 0 we use Theorem 2 in [6] to deduce that  $u \Psi'(u) / \Psi(u) \rightarrow 1 + \alpha$  as  $u \downarrow 0$ . Hence we get

$$u(r, \infty)^\alpha L(u(r, \infty)) \sim \frac{1}{\alpha r} \text{ as } r \rightarrow \infty \quad (12)$$

Since  $r \mapsto u(r, \infty)$  decreases from  $\infty$  to 0, for any  $t \in (0, \infty)$  there exists  $s(t) = s \in (0, \infty)$  such that  $u(s, \infty) = \lambda f(t)$ . Since  $L$  is slowly varying and  $f(t) \sim u(t, \infty)$  as  $t \rightarrow \infty$ , we use (12) to deduce that  $\lambda^\alpha s \sim t$  as  $t \rightarrow \infty$ . Also notice that

$$\begin{aligned} \log \left( \frac{u(t+s, \infty)}{u(t, \infty)} \right) &= \int_t^{t+s} \frac{\partial_r u(r, \infty)}{u(r, \infty)} dr = - \int_t^{t+s} u(r, \infty)^\alpha L(u(r, \infty)) dr \\ &\underset{t \rightarrow \infty}{\sim} -\frac{1}{\alpha} \log(1 + \lambda^{-\alpha}) \end{aligned}$$

where we use (12) at second line. Then, we have

$$\begin{aligned} \mathbb{E}_x \left[ e^{-\lambda Z_t f(t)} \mid t < T \right] &= \frac{e^{-x u(t, \lambda f(t))} - e^{-x u(t, \infty)}}{1 - e^{-x u(t, \infty)}} \\ &\underset{t \rightarrow \infty}{\sim} 1 - \frac{u(t+s, \infty)}{u(t, \infty)} \\ &\underset{t \rightarrow \infty}{\sim} 1 - (1 + \lambda^{-\alpha})^{-1/\alpha} \end{aligned}$$

This ends the proof. ■

## 4 Quasi-stationary distributions and Q-process in the explosive case

In this section, we consider a branching mechanism  $\Psi$  which verifies (5) so that  $Z$  explodes in finite time almost surely. An elementary calculation entails for all  $t \geq 0$  and all  $x > 0$

$$\mathbb{P}_x(T > t) = e^{-x u(t, 0+)}$$

We introduce

$$\Phi(\lambda) := \int_t^0 \frac{du}{\Psi(u)}, \quad \forall \lambda \geq 0$$

This increasing function admits a continuous inverse, namely the function  $t \mapsto u(t, 0+)$ . Also, thanks to Equation (11) we deduce the identities

$$\Phi(u(t, \lambda)) = t + \Phi(\lambda), \quad u(t, \lambda) = u(t + \Phi(\lambda), 0+), \quad \forall t, \lambda \geq 0 \quad (13)$$

Finally we introduce the quantity  $\Psi(+\infty) := \lim_{u \rightarrow +\infty} \Psi(u) \in [-\infty, 0)$ . The dominated convergence theorem ensures that  $\Psi(+\infty) = -\nu(0, \infty)$  as soon as  $\Psi(+\infty)$  is not infinite.



## 4.1 Proof of Theorem 1

First we compute the necessary form of the QSD. Fix  $\beta > 0$ . We get for all  $t \geq 0$

$$e^{-\beta t} = \mathbb{P}_{\mu_\beta}(T > t) = \int_{(0,\infty)} \mu_\beta(dr) e^{-r u(t,0+)}$$

Letting  $t = \Phi(\lambda)$  for any  $\lambda \geq 0$  we obtain

$$e^{-\beta\Phi(\lambda)} = \int_{(0,\infty)} \mu_\beta(dr) e^{-r\lambda}$$

Consequently there is at most one QSD corresponding to the rate of decay  $\beta$ . Now suppose that  $\mu_\beta$  is a probability distribution on  $(0, \infty)$  then the following calculation ensures that it is quasi-stationary. For any  $\lambda \in (0, \infty)$  we have

$$\begin{aligned} \mathbb{E}_{\mu_\beta}[e^{-\lambda Z_t} | T > t] &= \frac{\mathbb{E}_{\mu_\beta}[e^{-\lambda Z_t}; T > t]}{\mathbb{P}_{\mu_\beta}(T > t)} = \frac{\mathbb{E}_{\mu_\beta}[e^{-\lambda Z_t}]}{\mathbb{P}_{\mu_\beta}(T > t)} \\ &= \frac{\int_{(0,\infty)} \mu_\beta(dr) e^{-r u(t,\lambda)}}{\int_{(0,\infty)} \mu_\beta(dr) e^{-r u(t,0+)}} = \frac{e^{-\beta\Phi(u(t,\lambda))}}{e^{-\beta\Phi(u(t,0+))}} \\ &= e^{-\beta\Phi(\lambda)} = \mathbb{E}_{\mu_\beta}[e^{-\lambda Z_0}] \end{aligned}$$

We now assume  $\Psi(+\infty) \in (-\infty, 0)$  and we prove that  $\lambda \mapsto e^{-\beta\Phi(\lambda)}$  is indeed the Laplace transform of a probability measure  $\mu_\beta$  on  $(0, \infty)$ . Let  $x := \beta/\nu(0, \infty)$ , for all  $\lambda > 0$  we have

$$\mathbb{E}_x[e^{-\lambda Z_t} | T > t] = \frac{\mathbb{E}_x[e^{-\lambda Z_t}; T > t]}{\mathbb{P}_x(T > t)} = \exp\left(-x(u(t + \Phi(\lambda), 0+) - u(t, 0+))\right)$$

From Equation (11) we get that  $u(t, 0+) \rightarrow \infty$  as  $t \rightarrow \infty$  and that

$$\int_{u(t+\Phi(\lambda), 0+)}^{u(t, 0+)} \frac{du}{\Psi(u)} = \Phi(\lambda)$$

Since  $\Psi(u) \rightarrow -\nu(0, \infty)$  as  $u \rightarrow \infty$ , one deduces that

$$\int_{u(t+\Phi(\lambda), 0+)}^{u(t, 0+)} \frac{du}{\Psi(u)} \underset{t \rightarrow \infty}{\sim} \frac{u(t + \Phi(\lambda), 0+) - u(t, 0+)}{\nu(0, \infty)}$$

and therefore

$$\mathbb{E}_x[e^{-\lambda Z_t} | T > t] \xrightarrow[t \rightarrow \infty]{} e^{-\Phi(\lambda) x \nu(0, \infty)}$$

Since  $\Phi(\lambda) \rightarrow 0$  as  $\lambda \downarrow 0$ , we deduce that  $\lambda \mapsto e^{-\Phi(\lambda) x \nu(0, \infty)}$  is the Laplace transform of a probability measure on  $[0, \infty)$ . Moreover, it does not charge 0 since  $\Phi(\lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ .

We now suppose  $\Psi(-\infty) = -\infty$ . An easy adaptation of the preceding arguments ensure that for any  $x, \lambda > 0$

$$\mathbb{E}_x[e^{-\lambda Z_t} | T > t] \xrightarrow[t \rightarrow \infty]{} 0$$

Hence the limiting distribution is trivial: it is a Dirac mass at infinity. However, let us prove that  $\mu_\beta$  is a well-defined probability distribution on  $(0, \infty)$ . For every  $\epsilon > 0$ , define the branching mechanism

$$\begin{aligned}\Psi_\epsilon(u) &:= \int_{(\epsilon, \infty)} (e^{-hu} - 1)(\nu(dh) + \frac{1}{\epsilon}\delta_{-D\epsilon}(dh)) \\ &= \frac{1}{\epsilon}(e^{D\epsilon u} - 1) + \int_{(\epsilon, \infty)} (e^{-hu} - 1)\nu(dh)\end{aligned}$$

Observe that for any  $u \geq 0$ ,  $\Psi_\epsilon(u) \downarrow \Psi(u)$  as  $\epsilon \downarrow 0$ . Thus by monotone convergence we deduce that

$$\int_\lambda^0 \frac{du}{\Psi_\epsilon(u)} \xrightarrow{\epsilon \downarrow 0} \int_\lambda^0 \frac{du}{\Psi(u)}, \quad \forall \lambda \geq 0$$

The first part of the proof applies to  $\Psi_\epsilon$ , and therefore the l.h.s. of the preceding equation is the Laplace transform taken at  $\lambda$  of an infinitely divisible distribution on  $(0, \infty)$ . Since the limit is continuous at 0 and vanishes at  $\infty$ , we deduce that it is the Laplace transform of an infinitely divisible distribution on  $(0, \infty)$ . The asserted property follows.  $\blacksquare$

## 4.2 Proof of Theorem 2

Fix  $\lambda, x \in (0, \infty)$ . For any  $t \in (0, \infty)$ , we have

$$-\frac{1}{x} \log \mathbb{E}_x[e^{-\lambda Z_t/f(t)} \mid t < T] = u(t, \lambda f(t)^{-1}) - u(t, 0+)$$

The proof relies on two lemmas, whose proofs are postponed to the end of the subsection.

**Lemma 4.1** *As  $u \downarrow 0$ , we have  $\Phi(u) \sim u/(-\alpha\Psi(u))$ .*

Observe that  $f(t) \rightarrow +\infty$  as  $t \rightarrow \infty$ . The lemma now implies

$$\Psi(u(t, 0+))\Phi(\lambda f(t)^{-1}) \underset{t \rightarrow \infty}{\sim} -\frac{\Psi(u(t, 0+))\lambda}{\alpha f(t)\Psi(\lambda f(t)^{-1})}$$

Since  $L$  is slowly varying at  $0+$ , we deduce that  $\Psi(\lambda f(t)^{-1}) \sim \lambda^{1-\alpha}\Psi(f(t)^{-1})$  as  $t \rightarrow \infty$ . Thus the very definition of  $f$  entails

$$\Psi(u(t, 0+))\Phi(\lambda f(t)^{-1}) \underset{t \rightarrow \infty}{\sim} -\lambda^\alpha \alpha^{-1} \quad (14)$$

**Lemma 4.2** *The following holds true as  $t \rightarrow \infty$*

$$\int_{u(t, \lambda f(t)^{-1})}^{u(t, 0+)} \frac{dv}{\Psi(v)} \sim \int_{u(t, \lambda f(t)^{-1})}^{u(t, 0+)} \frac{dv}{\Psi(u(t, 0+))}$$

From the latter lemma, we deduce

$$\begin{aligned}u(t, \lambda f(t)^{-1}) - u(t, 0+) &\underset{t \rightarrow \infty}{\sim} -\Psi(u(t, 0+)) \int_{u(t, \lambda f(t)^{-1})}^{u(t, 0+)} \frac{dv}{\Psi(v)} \\ &\underset{t \rightarrow \infty}{\sim} -\Psi(u(t, 0+))\Phi(\lambda f(t)^{-1}) \\ &\underset{t \rightarrow \infty}{\sim} \lambda^\alpha \alpha^{-1}\end{aligned}$$

where we use Equations (11) and (14) at the second and third line. The theorem is proved.  $\blacksquare$

*Proof of Lemma 4.1.* Recall the definition of  $\Phi$ . An integration by parts yields that for all  $u \in [0, \infty)$

$$\Phi(u) = -\frac{u}{\Psi(u)} + \int_u^0 \frac{1}{\Psi(v)} \frac{v\Psi'(v)}{\Psi(v)} dv$$

Recall from Theorem 2 in [6] that  $v\Psi'(v)/\Psi(v) \rightarrow 1 - \alpha$  as  $v \downarrow 0$ . Therefore an elementary calculation ends the proof.  $\blacksquare$

*Proof of Lemma 4.2.* For all  $t \in [0, \infty)$  we have

$$\int_{u(t, \lambda f(t)^{-1})}^{u(t, 0+)} \frac{dv}{\Psi(v)} - \int_{u(t, \lambda f(t)^{-1})}^{u(t, 0+)} \frac{dv}{\Psi(u(t, 0+))} = \int_{u(t, \lambda f(t)^{-1})}^{u(t, 0+)} \frac{\Psi(u(t, 0+)) - \Psi(v)}{\Psi(v)\Psi(u(t, 0+))} dv$$

The convexity of  $\Psi$  implies that for all  $v \in [u(t, 0+), u(t, \lambda f(t)^{-1})]$  we have

$$0 \leq \Psi(u(t, 0+)) - \Psi(v) \leq -\Psi'(u(t, 0+))(v - u(t, 0+)) \quad (15)$$

Since  $\Psi'(v)/\Psi(v)$  goes to 0 as  $v \rightarrow \infty$ , the following identity (and therefore the lemma)

$$\int_{u(t, \lambda f(t)^{-1})}^{u(t, 0+)} \frac{\Psi(u(t, 0+)) - \Psi(v)}{\Psi(v)\Psi(u(t, 0+))} dv \underset{t \rightarrow \infty}{=} o\left(\int_{u(t, \lambda f(t)^{-1})}^{u(t, 0+)} \frac{dv}{\Psi(v)}\right)$$

would be ensured by (15) and the boundedness of  $t \mapsto u(t, \lambda f(t)^{-1}) - u(t, 0+)$ , that we now prove. Fix  $k \in (-2D, \infty)$ . Since  $\Psi'(v)$  converges to  $D$  as  $v \rightarrow \infty$ , for  $t$  large enough we get from (15) that  $\Psi(v) \geq \Psi(u(t, 0+)) - k(v - u(t, 0+))$  for all  $v \in [u(t, 0+), u(t, \lambda f(t)^{-1})]$ . A simple calculation then yields

$$0 \leq \frac{1}{k} \log \left(1 - k \frac{u(t, \lambda f(t)^{-1}) - u(t, 0+)}{\Psi(u(t, 0+))}\right) \leq \int_{u(t, \lambda f(t)^{-1})}^{u(t, 0+)} \frac{dv}{\Psi(v)} = \Phi(\lambda f(t)^{-1})$$

Using  $\log(1 + v) \geq v/2$  for  $v$  small, and since  $\Phi(\lambda f(t)^{-1}) \rightarrow 0$ , we get for  $t$  large enough

$$0 \leq -\frac{u(t, \lambda f(t)^{-1}) - u(t, 0+)}{2\Psi(u(t, 0+))} \leq \Phi(\lambda f(t)^{-1})$$

From Equation (14), we deduce that  $t \mapsto u(t, \lambda f(t)^{-1}) - u(t, 0+)$  is bounded.  $\blacksquare$

### 4.3 Proof of Theorem 3

Fix  $t \geq 0$ . Since we are dealing with non-decreasing processes and since the asserted limiting process is continuous, the convergence of finite-dimensional marginals suffices to prove the theorem. Hence we will prove that for any  $n \geq 1$ , any  $n$ -uplets  $0 \leq t_1 \leq \dots \leq t_n \leq t$  and any coefficients  $\lambda_1, \dots, \lambda_n > 0$  we have

$$\lim_{s \rightarrow \infty} -\frac{1}{x} \log (\mathbb{E}_x[e^{-\lambda_1 Z_{t_1} - \dots - \lambda_n Z_{t_n}} | T > t + s]) = \lambda_1 d_{t_1} + \dots + \lambda_n d_{t_n} \quad (16)$$

Thanks to an easy recursion, we get

$$\begin{aligned} & -\frac{1}{x} \log (\mathbb{E}_x[e^{-\lambda_1 Z_{t_1} - \dots - \lambda_n Z_{t_n}} | T > t + s]) \\ &= u\left(t_1, \lambda_1 + u\left(t_2 - t_1, \lambda_2 + \dots + u\left(t_n - t_{n-1}, \lambda_n + u(t + s - t_n, 0+)\right) \dots\right)\right) - u(t + s, 0+) \end{aligned}$$

To prove (16), we proceed via a recurrence on  $n$ . We check the case  $n = 1$ . We know that  $\lambda \rightarrow u(t, \lambda)$  is a concave function from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  such that

$$\frac{u(t, \lambda)}{\lambda} \rightarrow d_t \text{ as } \lambda \rightarrow \infty$$

The concavity of  $\lambda \rightarrow u(t, \lambda)$  then implies that  $\partial_\lambda u(t, \lambda) \rightarrow d_t$  as  $\lambda \rightarrow \infty$ . Writing  $u(t + s, 0+) = u(t_1, u(t + s - t_1, 0+))$  one deduces that

$$u(t_1, \lambda_1 + u(t + s - t_1, 0+)) - u(t + s, 0+) \rightarrow \lambda_1 d_{t_1} \text{ as } s \rightarrow \infty$$

The result for  $n = 1$  follows. The end of the recursion is elementary. ■

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